

Uniqueness Cases in Odd Type Groups of Finite Morley Rank, Revisited

Alexandre V. Borovik

School of Mathematics, The University of Manchester
Oxford Road, Manchester M13 9PL, England
alexandre.borovik >>>at<<< gmail.com

Jeffrey Burdges

School of Mathematics and Statistics
University of St Andrews
Mathematical Institute
North Haugh, St Andrews KY16 9SS Scotland
burdges >>>at<<< gmail.com

Ali Nesin

Mathematics Department, Istanbul Bilgi University
Kuştepe Şişli, Istanbul, Turkey
anesin >>>at<<< bilgi.edu.tr

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Abstract

The paper contains versions of the Strong Embedding Theorem and the Uniqueness Subgroup Theorem for groups of finite Morley rank and odd type which are needed for the study of permutations actions and modules in the finite Morley rank category.

1 Introduction

This paper relates to the programme of study of linear and permutation actions of finite Morley rank started by Borovik and Cherlin in [5] and continued in [9]. The paper contributes important general structural results, the Strong Embedding Theorem and the Uniqueness Subgroup Theorem stated below, to the forthcoming paper by Berkman and Borovik [2] about the pseudoreflexion actions of groups of finite Morley rank.

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Strong Embedding Theorem. *Let G be a simple group of finite Morley rank and odd type with Prüfer 2-rank ≥ 3 . Let S be a Sylow 2-subgroup of G and $T = S^\circ$ the maximal 2-torus in S . Assume in addition that every proper definable subgroup of G containing T is a K -group.*

Then if G has a proper 2-generated core $M = \Gamma_{S,2}(G) < G$ then M is strongly embedded.

The Strong Embedding Theorem is a variation on the theme of an earlier version of a similar result under the same name [4].

Uniqueness Subgroup Theorem. *Let G be a simple group of finite Morley rank and odd type with Prüfer 2-rank ≥ 3 . Let S be a Sylow 2-subgroup of G and $D = S^\circ$ the maximal 2-torus in S . Let further E be the subgroup of D generated by involutions and θ a nilpotent signaliser functor on G .*

Assume in addition that every proper definable subgroup of G containing T is a K -group.

Then $M = N_G(\theta(E))$ is a strongly embedded subgroup in G .

An early version of the Uniqueness Subgroup Theorem was published as Theorem 6.7 in [3].

The notions of both 2-generated core and strongly embedded subgroup arise as so-called *uniqueness cases* in finite group theory. These subgroups both exhibit a black hole property reminiscent of a normal subgroup; strong embedding, however, is far more powerful and has global consequences which deeply affect the structure of the group.

The standard references for our basic facts and terminology are [6] and [1]. Here we give definitions immediately involved in the statement of the Strong Embedding Theorem.

A group G of finite Morley rank is *connected* if it contains no proper definable subgroup of finite index; the connected component G° of G is the maximal connected subgroup of G . The definable closure $d(X)$ of a subset $X \subseteq G$ is a minimal definable subgroup containing X . If $H \leq G$ is an arbitrary (not necessarily definable) subgroup, we set $H^\circ = H \cap d(H)^\circ$; this is a subgroup of finite index in H .

We define the 2-rank $m_2(G)$ of a group G to be the maximum rank of its elementary abelian 2-subgroups. Also, the Prüfer 2-rank $\text{pr}_2(G)$ is the maximum 2-rank of its Prüfer 2-subgroups $\mathbb{Z}_{2^\infty}^k$. These ranks must all be finite for subgroups of an odd type group of finite Morley rank.

Let G be a group of finite Morley rank and let S be a Sylow 2-subgroup of G . We define the 2-generated core $\Gamma_{S,2}(G)$ of G to be the definable hull of the group generated by all normalizers $N_G(U)$ of all elementary abelian 2-subgroups $U \leq S$ with $m_2(U) \geq 2$.

A proper definable subgroup M of a group G of finite Morley rank is said to be *strongly embedded* if M contains an involution but subgroups $M \cap M^g$ do not contain involutions for any $g \in G \setminus M$.

Our next group of definition concerns signaliser functors. For any involution $s \in G$, let $\theta(s) \leq O(C_G(s))$ be some connected definable normal subgroup of

$C_G(s)$. We say that θ is a *signalizer functor* if, for any commuting involutions $t, s \in G$,

$$\theta(t) \cap C_G(s) = \theta(s) \cap C_G(t).$$

The signalizer functor θ is *complete* if for any elementary abelian subgroup $E \leq G$ of order at least 8 the subgroup

$$\theta(E) = \langle \theta(t) \mid t \in E^\# \rangle$$

is a connected subgroup without 2-torsion and

$$C_{\theta(E)}(s) = \theta(s)$$

for any $s \in E^\#$. Finally, a signalizer functor θ is *non-trivial* if $\theta(s) \neq 1$ for some involution $s \in G$, and *nilpotent* if all the subgroups $\theta(t)$ are nilpotent.

2 Sylow Theory

Fact 2.1 ([7]; Lemma 10.8 of [6]). *Let S be a Sylow 2-subgroup of a group of finite Morley rank. Then $S^\circ = B * T$ is a central product of a definable connected nilpotent subgroup B of bounded exponent and of a 2-torus T , i.e. T is a divisible abelian 2-group.*

A group is said to have *odd type* if $B = 1$ and $T \neq 1$. This notion is well-defined because groups of finite Morley rank have a good theory of Sylow 2-subgroups:

Fact 2.2 (Theorem 10.11 of [6]). *The Sylow 2-subgroups of a group of finite Morley rank are conjugate.*

We will find the following corollary, known as a “Fratini argument”, to be useful.

Fact 2.3 (Corollary 10.12 of [6]). *Let G be a group of finite Morley rank, let $N \triangleleft G$ be a definable subgroup, and let S be a characteristic subgroup of the Sylow 2-subgroup of N . Then $G = N_G(S)N$.*

The key result describing conjugation patterns of 2-elements in a group of finite Morley rank is the following theorem by Burdges and Cherlin.

Fact 2.4 (Theorem 3 of [8]). *Let G be a connected group of finite Morley rank, and a any 2-element of G such that $C_G(a)$ is of odd type. Then a belongs to a 2-torus.*

Fact 2.5 (Lemma 4.5 of [3]). *If S is a 2-subgroup in a group of finite Morley rank and odd type with $\text{pr}_2(S) \geq 3$, then for any two four-subgroups (i.e. elementary abelian subgroups of order 4) $U, V \leq S$ there is a sequence of four-subgroups*

$$U = V_1, V_2, \dots, V_n = V,$$

such that $[V_i, V_{i+1}] = 1$ for all $i = 1, 2, \dots, n-1$.

If conclusions of fact 2.5 hold in a 2-group S , we say that S is *2-connected*.

3 The Generation Principle for K -groups

A group G of finite Morley rank is called a K -group if every non-trivial connected definable simple section of G is a Chevalley group over an algebraically closed field. In particular, nilpotent and solvable groups of finite Morley rank are K -groups.

A *Klein four-group*, or just *four-group* for short, is a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We will use the notation $H^\# = H \setminus \{1\}$ to denote the set of non-identity elements of a group H .

We need the following generation principle for K -groups:

Fact 3.1 (Theorem 5.14 of [3]). *Let G be a connected K -group of finite Morley rank and odd type. Let V be a four-subgroup acting definably on G . Then*

$$G = \langle C_G^\circ(v) \mid v \in V^\# \rangle$$

Our next lemma is an easy consequence of Fact 3.1.

Lemma 3.2. *Let G be a simple group of finite Morley rank and odd type; assume, in addition, that centralisers of involutions in G are K -groups. Let S be a Sylow 2-subgroup of G and let $M = \Gamma_{S,2}(G)$ be the 2-generated core associated with S . Let A be an elementary abelian 2-subgroup of M with $m_2(A) \geq 3$. Then $C_G^\circ(a) \leq M$ for any $a \in A^\#$.*

Proof. Let $K = C_G(a)$, then $A \leq K$ and K is a K -group of odd type. Let A_1 be a four-subgroup of A disjoint from $\langle a \rangle$. By Fact 3.1, $K^\circ = \langle C_K^\circ(x) \mid x \in A_1^\# \rangle$. Now $C_K^\circ(x) \leq C_G(a, x)$ and $\langle a, x \rangle$ is a four-subgroup of S . Thus $K^\circ \leq M$. \square

4 Strong Embedding

$I(H)$ to denote the set of involutions of H .

We will apply the usual criteria for strong embedding:

Fact 4.1 (Theorem 10.19 of [6]). *Let G be a group of finite Morley rank with a proper definable subgroup M . Then the following are equivalent:*

1. M is a strongly embedded subgroup.
2. $I(M) \neq \emptyset$, $C_G(i) \leq M$ for every $i \in I(M)$, and $N_G(S) \leq M$ for every Sylow 2-subgroup of M .
3. $I(M) \neq \emptyset$ and $N_G(S) \leq M$ for every non-trivial 2-subgroup S of M .

5 Proof of the Strong Embedding Theorem

Let G be a simple group of finite Morley rank and odd type with Prüfer 2-rank ≥ 3 . Let S be a Sylow 2-subgroup of G and $T = S^\circ$ a maximal 2-torus in G . Suppose that G has a proper 2-generated core $M = \Gamma_{S,2}(G) < G$. Assume

in addition that every proper definable subgroup of G which contains T is a K -group.

Claim 5.1. *Let t be a 2-element in G . Then t belongs to a 2-torus and $C_G(t)$ is a K -group.*

Verification. By Fact 2.4, t belongs to a maximal 2-torus R of G . Since by Sylow's Theorem $R^g = T$ for some $g \in G$, we see that $T \geq C_G(t)^g$ and therefore $C_G(t)^g$ and hence $C_G(t)$ is a K -group. \diamond

Denote by E the subgroup of T generated by all involutions in T ; then $E \triangleleft S$ and E is an elementary abelian 2-group with $m_2(E) \geq 3$.

Claim 5.2. *For every $i \in I(S)$, $C_E(i)$ contains a four-group.*

Verification. Since E is normal in S , the involution i induces a linear transformation of the \mathbb{F}_2 -vector space E . Since $m_2(E) > 2$, the Jordan canonical form of i cannot consist of a single block, so there are at least two eigenvectors. Since the eigenvalues associated to these eigenvectors must have order 2, the eigenvalues must both be 1, as desired. \diamond

Claim 5.3. $C_G^\circ(i) \leq M$ for every $i \in I(M)$.

Verification. We may assume that $i \in I(S)$ by Fact 2.2. By Claim 5.2, there is a four-group $E_1 \leq E$ centralized by i . Thus either E or $\langle E_1, i \rangle$ is an elementary abelian 2-group of rank at least three which contains i . By Fact 3.2, $C_G^\circ(i) \leq M$. \diamond

Claim 5.4. *Let R be a maximal 2-torus in M . Then $N_G(R) \leq M$.*

Verification. Observe that $N_G(R) \leq N_G(E) \leq M$ by definition of the proper 2-generated core. \diamond

Claim 5.5. $C_G(i) \leq M$ for every involution $i \in M$.

Verification. By Claim 5.1, $C_G(i)$ contains a maximal 2-torus R of G . Obviously, $R \leq C_G^\circ(i) \leq M$ by Claim 5.3. Now by Claim 5.4 $N_G(R) \leq M$. Applying the Frattini argument to $C_G(i)$, we see that

$$C_G(i) = C_G^\circ(i)N_{C_G(i)}(R) \leq C_G^\circ(i)N_G(R) \leq M \cdot M = M.$$

\diamond

Claim 5.6. $N_G(S) \leq M$.

Verification. This follows from Claim 5.4 since $N_G(S) \leq N_G(T) \leq M$. \diamond

Claims 5.5 and 5.6 show that M satisfies the criterion for being a strongly embedded subgroups (Fact 4.1) which completes the proof of the theorem.

6 Proof of the Uniqueness Subgroup Theorem

We start the proof by quoting a theorem on completeness of nilpotent signaliser functors.

Theorem 6.1. [6, Theorem B.30] *Any nilpotent signaliser functor θ on a group G of finite Morley rank is complete.*

Now let G be a group of odd type of Prüfer 2-rank at least 3 and S a Sylow 2-subgroup in G . Let T be a maximal 2-torus in S and E the subgroup generated by involutions in T ; obviously, E is a normal in S elementary abelian subgroup of order at least 8. Let θ be a non-trivial nilpotent signaliser functor on G . If $s \in S$ is any involution and $\theta(s) \neq 1$, then let us take $D = C_E(s)$; then $|D| \geq 4$. Obviously D normalizes $\theta(s)$ and

$$\theta(s) = \langle \theta(s) \cap C_t \mid t \in D^\# \rangle \leq \theta(E)$$

by Theorem 3.1. So, if we assume that $\theta(s) \neq 1$ for some involution $s \in P$, then we have $\theta(E) \neq 1$.

Now we can state our main result about signaliser 2-functors.

Theorem 6.2. *In the notation above, if a group G of finite Morley rank and odd type has Prüfer 2-rank at least 3 and admits a non-trivial nilpotent signaliser functor θ , then*

$$\Gamma_{S,2}(G) \leq N_G(\theta(E)).$$

In particular, G has a proper 2-generated core.

Now application of the Strongly Embedding Theorem immediately yields the main result of this paper:

Uniqueness Subgroup Theorem. *Let G be a simple group of finite Morley rank and odd type with Prüfer 2-rank ≥ 3 . Let S be a Sylow 2-subgroup of G and $D = S^\circ$ the maximal 2-torus in S . Let further E be the subgroup of D generated by involutions and θ a nilpotent signaliser functor on G .*

Assume in addition that every proper definable subgroup of G containing T is a K -group.

Then $M = N_G(\theta(E))$ is a strongly embedded subgroup in G .

Proof of Theorem 6.2. For an arbitrary four-subgroup $W < S$ write

$$\theta(W) = \langle \theta(w) \mid w \in W^\# \rangle.$$

Notice that, if V and W are two commuting four-subgroups in S , $[V, W] = 1$, and $v \in V^\#$, then $\theta(v)$ is W -invariant and, by Theorem 3.1,

$$\begin{aligned} \theta(v) &= \langle \theta(v) \cap C_w \mid w \in W^\# \rangle \\ &\leq \langle \theta(w) \mid w \in W^\# \rangle \\ &= \theta(W). \end{aligned}$$

Therefore $\theta(V) \leq \theta(W)$. Analogously $\theta(W) \leq \theta(V)$ and $\theta(V) = \theta(W)$.

Recall that the Sylow subgroup S is 2-connected by Lemma 2.5; this means that if U is a four-subgroup from E and V is an arbitrary four-subgroup in S then there is a sequence of four-subgroups

$$U = V_0, V_1, \dots, V_n = V$$

such that $[V_i, V_{i+1}] = 1$ for $i = 0, 1, \dots, n-1$. Therefore, by the previous paragraph, $\theta(V) = \theta(U)$. But

$$\begin{aligned} \theta(U) &= \langle \theta(u) \mid u \in U^\# \rangle \\ &= \langle C_{\theta(E)}(u) \mid u \in U^\# \rangle \\ &= \theta(E), \end{aligned}$$

by Theorem 3.1. Therefore $\theta(V) = \theta(E)$ for any four-subgroup $V < S$.

If now $P \leq S$ is a subgroup of 2-rank at least 2, $g \in N_G(P)$ and $V \leq P$ is a four-subgroup, we have

$$\theta(E)^g = \theta(V)^g = \theta(V^g) = \theta(E)$$

and $N_G(P) \leq N_G(\theta(E))$. Therefore $\Gamma_{S,2}(G) \leq N_G(\theta(E))$. \square

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